JOURNAL OF COMPUTATIONAL PHYSICS 99, 310–320 (1992)

Simulation of the Steady-State Energy Transfer in Rigid Bodies, with Convective/Radiative Boundary Conditions, Employing a Minimum Principle

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Received June 7, 1990; revised May 9, 1991

The subject of this paper is the energy transfer phenomenon in a rigid and opaque body that exchanges energy, with the environment, by convection and by diffuse thermal radiation. The considered phenomenon is described by a partial differential equation, subjected to (nonlinear) boundary conditions. It is presented with a minimum principle, suitable for a large class of energy transfer problems. Some particular cases are simulated. © 1992 Academic Press, Inc.

INTRODUCTION

The majority of the works whose main subject is the energy transfer phenomenon in rigid bodies is concerned with linear descriptions. In other words, the phenomena are usually represented by a linear mathematical problem.

However, in general, the reality is not linear and, in many situations, cannot be approximated by linear descriptions. One of these situations occurs, for instance, when a body is at very high temperature and, consequently, the heat loss by thermal radiation is not negligible.

The main subject of this work is the coupled conduction/convection/radiation heat transfer phenomenon in a rigid and opaque body with internal heat supply.

The main objective is to present a reliable way for simulating such nonlinear problems using finite elements. In other words, it will be presented with a minimum principle suitable for a large class of nonlinear energy transfer problems.

This minimum principle will be represented by a continuous, convex, and coercive functional whose existence provides an useful way for numerical simulations and assures the solution's existence and uniqueness.

The use of the minimum principle in the simulation of nonlinear energy transfer phenomena will be illustrated in this work by a typical example in which the conduction/convection/radiation heat transfer phenomenon in a cylindrical body is simulated using finite elements.

THE STEADY-STATE HEAT TRANSFER PHENOMENON

Let us consider a continuous body $\mathfrak{B}$ represented by the bounded open set Ω with regular boundary $\partial\Omega$. When this body is assumed rigid and opaque the energy transfer mechanism inside Ω is the conduction heat transfer. The steady-state conduction heat transfer phenomenon is governed by the equation [1]

$$-\operatorname{Div} \mathbf{q} + q''' = 0 \qquad \text{in } \Omega \tag{1}$$

in which q''' represents the internal heat supply, per unit time and unit volume, and $\mathbf{q}$ is given by (Fourier Law)

$$\mathbf{q} = -\mathbf{K} \operatorname{Grad} T. \tag{2}$$

The fields $\mathbf{q}$, T, and $\mathbf{K}$ represent, respectively, the conduction heat flux (per unit time and unit area), the absolute temperature, and the thermal conductivity. The tensor field $\mathbf{K}$ is positive-definite and, in this paper, may depend only on the position $\mathbf{X}$.

The above equation (energy balance) must be subjected to boundary conditions. These boundary conditions arise naturally when it is imposed continuity in the normal heat flux across $\partial\Omega$. Continuity holds if the normal conduction heat flux is equal to the sum of radiant and convective heat fluxes on $\partial\Omega$. This condition may be represented as follows

$$\mathbf{q} \circ \mathbf{n} = q_{\text{conv}} + q_{\text{rad}} \qquad \text{on } \partial\Omega \tag{3}$$

in which $\mathbf{n}$ is the unit outward normal (defined for all $\mathbf{X} \in \partial\Omega$), q_{conv} is the heat (per unit time and unit area) lost by convection, and q_{rad} is the heat (per unit time and unit area) lost by thermal radiation.

Usually the convective heat loss is given by [2]

$$q_{\text{conv}} = h(T - T_\infty) \qquad \text{on } \partial\Omega \tag{4}$$

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in which T_∞ is a temperature of reference ($T_\infty = \hat{T}_\infty(\mathbf{X}) \geqslant 0$, $\mathbf{X} \in \partial\Omega$) and h is the convection heat transfer coefficient field ($h \geqslant 0$).

The radiative heat loss is given by [3]

$$q_{\text{rad}} = \varepsilon\sigma T^4 - c \qquad \text{on } \partial\Omega \tag{5}$$

in which σ is the Stefan–Boltzmann constant, ε is the emittance field ($0 < \varepsilon \leqslant 1$) and c ($c \geqslant 0$) is a given field that represents the thermal radiant energy coming from the environment.

Combining (1), (2), (3), (4), and (5) we have the mathematical problem

$$\begin{aligned} &\operatorname{Div}(\mathbf{K} \operatorname{Grad} T) + q''' = 0 \qquad \text{in } \Omega \\ &-\mathbf{K} \operatorname{Grad} T \circ \mathbf{n} = \varepsilon\sigma T^4 - c + h(T - T_\infty) \\ &\qquad \text{on } \partial\Omega \end{aligned} \tag{6}$$

in which the unknown is the absolute temperature T.

Since T represents an absolute temperature we shall employ, instead of Eq. (5), the equation

$$q_{\text{rad}} = \varepsilon\sigma\, |T|^3\, T - c \qquad \text{on } \partial\Omega \tag{7}$$

which is thermodynamically equivalent (because $T \geqslant 0$) and mathematically more convenient.

The employment of Eq. (7) is fundamental for the existence of a minimum principle.

Combining (1), (2), (3), (4), and (7) we have the mathematical problem

$$\begin{aligned} &\operatorname{Div}(\mathbf{K} \operatorname{Grad} T) + q''' = 0 \qquad \text{in } \Omega \\ &-\mathbf{K} \operatorname{Grad} T \circ \mathbf{n} = \varepsilon\sigma\, |T|^3\, T - c + h(T - T_\infty) \\ &\qquad \text{on } \partial\Omega \end{aligned} \tag{8}$$

in which the unknown is the absolute temperature T.

In order to enlarge the class of phenomena to be studied we shall discuss, in the next section, a generalization of problem (8) and its variational formulation.

A GENERALIZED HEAT TRANSFER PROBLEM AND ITS MINIMUM PRINCIPLE

Let us consider the problem (that may be regarded as a generalization of Eq. (8))

$$\begin{aligned} \operatorname{Div}(\mathbf{K} \operatorname{Grad} T) + q''' &= 0 \qquad \text{in } \Omega \\ -\mathbf{K} \operatorname{Grad} T \circ \mathbf{n} &= f \qquad \text{on } \partial\Omega \end{aligned} \tag{9}$$

in which the fields f ($f = \hat{f}(T, \mathbf{X})$, $\mathbf{X} \in \partial\Omega$) and q''' ($q''' = \hat{q}'''(T, \mathbf{X})$, $\mathbf{X} \in \Omega$) may depend on the unknown T, provided

$$\hat{q}'''(T_1, \mathbf{X}) \geqslant \hat{q}'''(T_2, \mathbf{X}) \qquad \text{if} \quad T_1 < T_2 \quad \text{for all} \quad \mathbf{X} \in \Omega \tag{10}$$

$$\hat{f}(T_1, \mathbf{X}) \leqslant \hat{f}(T_2, \mathbf{X}) \qquad \text{if} \quad T_1 < T_2 \quad \text{for all} \quad \mathbf{X} \in \partial\Omega \tag{11}$$

hold and provided there exists an open nonempty subset $\partial\Omega^+$ ($\partial\Omega^+ \subseteq \partial\Omega$) such that

$$\hat{f}(T_1, \mathbf{X}) < \hat{f}(T_2, \mathbf{X}) \qquad \text{if} \quad T_1 < T_2 \quad \text{for all} \quad \mathbf{X} \in \partial\Omega^+ \tag{12}$$

$$\lim_{T \to \infty} \hat{f}(T, \mathbf{X}) = +\infty \qquad \text{for all} \quad \mathbf{X} \in \partial\Omega^+ \tag{13}$$

$$\lim_{T \to -\infty} \hat{f}(T, \mathbf{X}) = -\infty \qquad \text{for all} \quad \mathbf{X} \in \partial\Omega^+. \tag{14}$$

PROPOSITION. *The solution of problem* (9) *exists, is unique, and is the field which minimizes the functional*

$$I[u] = \tfrac{1}{2} \int_\Omega \{\operatorname{Grad} u \circ \mathbf{K} \operatorname{Grad} u\}\, dV - \int_\Omega Q\, dV + \int_{\partial\Omega} F\, dS \tag{15}$$

in which

$$Q = \hat{Q}(u, \mathbf{X}) = \int_0^u \hat{q}'''(y, \mathbf{X})\, dy \tag{16}$$

$$F = \hat{F}(u, \mathbf{X}) = \int_0^u \hat{f}(y, \mathbf{X})\, dy. \tag{17}$$

Proof. The proof will be divided into three parts:

(1) *System* (9) *represents the Euler–Lagrange equation and the natural boundary conditions associated to* $I[u]$. Let us consider the following admissible fields

$$u = T + \alpha\eta, \qquad u \in H^1(\Omega), \tag{18}$$

in which T is a solution of (9), α is a real-valued parameter and η is an admissible variation.

Hence, the first variation of functional $I[u]$, defined in (15), is given as

$$\delta I = \int_\Omega \{\operatorname{Grad} \eta \circ \mathbf{K} \operatorname{Grad} T - \hat{q}'''(T, \mathbf{X})\, \eta\}\, dV + \int_{\partial\Omega} \hat{f}(T, \mathbf{X})\, \eta\, dS. \tag{19}$$

Employing Green's Identity and imposing $\delta I = 0$ we obtain (since η is arbitrary)

$$\int_\Omega \{-\operatorname{Div}(\mathbf{K}\operatorname{Grad} T) - q'''\}\,\eta\, dV = 0 \tag{20}$$

$$\int_\Omega \{\mathbf{K}\operatorname{Grad} T \circ \mathbf{n} + f\}\,\eta\, dS = 0 \tag{21}$$

which represents (9), in a weak sense. It is to be noticed that (assuming that q''' belongs to $L^2(\Omega)$ and that $\mathbf{K}$ belongs to $C^{0,1}(\bar{\Omega})$), T belongs to $H^2(\Omega)$ and, hence, since $\Omega \subset \mathbb{R}^3$ and Ω is sufficiently regular, T is continuous in Ω [4].

(2) *$\delta I = 0$ corresponds to a minimum, which is unique.* This assertion holds if I is a strictly convex functional; that means

$$I[\theta T_1 + (1-\theta)T_2] < \theta I[T_1] + (1-\theta) I[T_2], \qquad T_1 \not\equiv T_2 \quad \text{in } \bar{\Omega},\ \theta \in (0,1). \tag{22}$$

The above inequality is equivalent to

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \{\operatorname{Grad}[\theta T_1 + (1-\theta)T_2] \\
&\qquad \circ \mathbf{K}\operatorname{Grad}[\theta T_1 + (1-\theta)T_2]\}\, dV \\
&\qquad - \int_\Omega \hat{Q}(\theta T_1 + (1-\theta)T_2, \mathbf{X})\, dV \\
&\qquad + \int_{\partial\Omega} \hat{F}(\theta T_1 + (1-\theta)T_2, \mathbf{X})\, dS \\
&< \tfrac{1}{2}\int_\Omega \{\theta \operatorname{Grad} T_1 \circ \mathbf{K}\operatorname{Grad} T_1 \\
&\qquad + (1-\theta)\operatorname{Grad} T_2 \circ \mathbf{K}\operatorname{Grad} T_2\}\, dV \\
&\qquad - \int_\Omega \{\theta \hat{Q}(T_1, \mathbf{X}) + (1-\theta)\hat{Q}(T_2, \mathbf{X})\}\, dV \\
&\qquad + \int_{\partial\Omega} \{\theta \hat{F}(T_1, \mathbf{X}) + (1-\theta)\hat{F}(T_2, \mathbf{X})\}\, dS,
\end{aligned} \tag{23}$$

or

$$\begin{aligned}
&\tfrac{1}{2}\int_\Omega \theta(1-\theta)\operatorname{Grad}(T_1 - T_2) \circ \mathbf{K}\operatorname{Grad}(T_1 - T_2)\, dV \\
&\qquad + \int_{\partial\Omega} \{\theta \hat{F}(T_1, \mathbf{X}) + (1-\theta)\hat{F}(T_2, \mathbf{X}) \\
&\qquad - \hat{F}(\theta T_1 + (1-\theta)T_2, \mathbf{X})\}\, dS \\
&\qquad - \int_\Omega \{\theta \hat{Q}(T_1, \mathbf{X}) + (1-\theta)\hat{Q}(T_2, \mathbf{X}) \\
&\qquad - \hat{Q}(\theta T_1 + (1-\theta)T_2, \mathbf{X})\}\, dV > 0.
\end{aligned} \tag{24}$$

Since (10), (11), and (12) hold we have

$$\begin{aligned}
&\theta \hat{F}(T_1, \mathbf{X}) + (1-\theta)\hat{F}(T_2, \mathbf{X}) - \hat{F}(\theta T_1 + (1-\theta)T_2, \mathbf{X}) \geqslant 0, \\
&\qquad T_1 \not\equiv T_2 \quad \text{on } \partial\Omega, \\
&\theta F(T_1, \mathbf{X}) + (1-\theta)F(T_2, \mathbf{X}) - F(\theta T_1 + (1-\theta)T_2, \mathbf{X}) > 0, \\
&\qquad T_1 \not\equiv T_2 \quad \text{on } \partial\Omega^+,\ \theta \in (0,1)
\end{aligned} \tag{25}$$

$$\begin{aligned}
&-\theta \hat{Q}(T_1, \mathbf{X}) - (1-\theta)\hat{Q}(T_2, \mathbf{X}) \\
&\qquad + \hat{Q}(\theta T_1 + (1-\theta)T_2, \mathbf{X}) \geqslant 0, \\
&\qquad T_1 \not\equiv T_2 \quad \text{in } \Omega,\ \theta \in (0,1).
\end{aligned} \tag{26}$$

Hence the left side of inequality (24) is always nonnegative.

Let us suppose that the left side of (24) becomes zero. In this case, once that $\mathbf{K}$ is a positive-definite tensorial field and $\partial\Omega^+$ is nonempty, we must have

$$T_1 \equiv T_2 + \text{const} \qquad \text{in } \Omega \tag{27}$$

$$T_1 \equiv T_2 \qquad \text{on } \partial\Omega. \tag{28}$$

In order to satisfy (27) and (28) we must have

$$T_1 \equiv T_2 \qquad \text{in } \bar{\Omega}. \tag{29}$$

Hence the left side of (24) never becomes zero for $T_1 \not\equiv T_2$ in $\bar{\Omega}$ and, therefore, I is a strictly convex functional.

With these results we conclude that $\delta I = 0$ corresponds to a minimum, which is unique.

(3) *The field T, which minimizes I, exists (solution's existence).* Now, in order to assure the solution's existence it is sufficient to show that I is a coercive functional [5]. A sufficient condition for coerciveness is

$$\lim_{\gamma \to \infty} \left(\frac{I[\gamma u]}{\gamma}\right) = +\infty, \qquad u \in H^1(\Omega),\ u \not\equiv 0. \tag{30}$$

Aiming to demonstrate that (30) holds, we begin taking into account (10) and (11). Since q''' and f satisfy (10) and (11) we conclude that

$$-\hat{Q}(u, \mathbf{X}) \geqslant \hat{\alpha}(\mathbf{X})u; \qquad \alpha = \hat{\alpha}(\mathbf{X}) \equiv -\hat{q}'''(0, \mathbf{X}), \qquad \mathbf{X} \in \Omega \tag{31}$$

$$\hat{F}(u, \mathbf{X}) \geqslant \hat{\beta}(\mathbf{X})u; \qquad \beta = \hat{\beta}(\mathbf{X}) \equiv \hat{f}(0, \mathbf{X}), \qquad \mathbf{X} \in \partial\Omega. \tag{32}$$

Hence, we may write

$$\begin{aligned}
I[\gamma u] \geqslant{}& \tfrac{1}{2}\int_\Omega \{\operatorname{Grad}\gamma u \circ \mathbf{K}\operatorname{Grad}\gamma u\}\, dV \\
&+ \int_{\partial\Omega^+} \hat{F}(\gamma u, \mathbf{X})\, dS \\
&+ \int_{\partial\Omega - \partial\Omega^+} \beta\gamma u\, dS + \int_\Omega \alpha\gamma u\, dV
\end{aligned} \tag{33}$$

and, consequently,

$$\frac{1}{\gamma} I[\gamma u] \geqslant \frac{\gamma}{2} \int_{\Omega} \{\text{Grad}\, u \circ \mathbf{K}\, \text{Grad}\, u\}\, dV + \int_{\partial\Omega^+} \frac{\hat{F}(\gamma u, \mathbf{X})}{\gamma}\, dS + \int_{\partial\Omega - \partial\Omega^+} \beta u\, dS + \int_{\Omega} \alpha u\, dV. \tag{34}$$

Since f satisfies (12), (13), and (14) we have that $\hat{F}(\gamma u, \mathbf{X})$ has, for any $\mathbf{X} \in \partial\Omega^+$, a lower bound. In other words,

$$\int_{\partial\Omega^+} \hat{F}(\gamma u, \mathbf{X})\, dS > \text{const.} \tag{35}$$

Hence, if u is not constant in $\bar{\Omega}$, we conclude that

$$\frac{1}{\gamma} I[\gamma u] \geqslant \frac{\gamma}{2} \int_{\Omega} \{\text{Grad}\, u \circ \mathbf{K}\, \text{Grad}\, u\}\, dV + \text{const.} \tag{36}$$

and, therefore,

$$\lim_{\gamma \to \infty} \left(\frac{I[\gamma u]}{\gamma}\right) \geqslant \lim_{\gamma \to \infty} \frac{\gamma}{2} \int_{\Omega} \{\text{Grad}\, u \circ \mathbf{K}\, \text{Grad}\, u\}\, dV + \text{const} = +\infty. \tag{37}$$

On the other hand, if $u \equiv \text{const}$ in $\bar{\Omega}$, we have that

$$\frac{1}{\gamma} I[\gamma u] \geqslant \int_{\partial\Omega^+} \frac{F(\gamma u, \mathbf{X})}{\gamma}\, dS + \text{const} \tag{38}$$

and, therefore,

$$\lim_{\gamma \to \infty} \frac{1}{\gamma} I[\gamma u] \geqslant \lim_{\gamma \to \infty} \int_{\partial\Omega^+} \frac{\hat{F}(\gamma u, \mathbf{X})}{\gamma}\, dS + \text{const.} \tag{39}$$

Since (13) and (14) hold, we have that

$$\lim_{\gamma \to \infty} \frac{\hat{F}(\gamma u, \mathbf{X})}{\gamma} = \lim_{\gamma \to \infty} \left(\hat{f}(\gamma u, \mathbf{X}) \frac{u}{|u|}\right) = +\infty, \qquad u \equiv \text{const.} \tag{40}$$

and, consequently, from (39),

$$\lim_{\gamma \to \infty} \left(\frac{I[\gamma u]}{\gamma}\right) = +\infty \qquad \text{for} \quad u \equiv \text{const in } \bar{\Omega}. \tag{41}$$

Thus, we have proven that (30) holds. This result, associated to the functional's continuity and convexity, assures the existence of a solution that, as we have demonstrated, is unique.

THE MINIMUM PRINCIPLE FOR AN IMPORTANT PARTICULAR CASE

Let us assume that (in problem (8)) ε, h, T_∞, c, and q''' depend only on the position $\mathbf{X}$ ($\varepsilon = \hat{\varepsilon}(\mathbf{X})$, $h = \hat{h}(\mathbf{X})$, $T_\infty = \hat{T}_\infty(\mathbf{X})$, $c = \hat{c}(\mathbf{X})$, and $q''' = \hat{q}'''(\mathbf{X})$). In such a situation the functional $I[u]$ is given as

$$I[u] = \tfrac{1}{2} \int_{\Omega} \{\text{Grad}\, u \circ \mathbf{K}\, \text{Grad}\, u\}\, dV - \int_{\Omega} q''' u\, dV + \int_{\partial\Omega} \{\tfrac{1}{5}\sigma\varepsilon\, |u|^5 + \tfrac{1}{2} h(u - T_\infty)^2 - cu\}\, dS. \tag{42}$$

Here the field q''' satisfies (10), once that does not depend on T, and the field f satisfies (11), (12), and (13), being given as

$$\hat{f}(T, \mathbf{X}) = \hat{\varepsilon}(\mathbf{X})\, |T|^3\, T + \hat{h}(\mathbf{X})(T - \hat{T}_\infty(\mathbf{X})) - \hat{c}(\mathbf{X}), \qquad \text{on } \partial\Omega. \tag{43}$$

It is to be noted that (11) is not satisfied if, instead of (7), we employ Eq. (5) for calculating the thermal radiant loss. The mathematical behavior of the field f was used for constructing the form presented in (7), which is physically equivalent to (5) while mathematically quite different.

SOME TYPICAL SITUATIONS DESCRIBED BY (9)

Several important situations may be described by (9). The most famous is the one which takes into account only the convective heat loss in a linear way [2, 6] giving rise to the following form of f:

$$f = h(T - T_\infty), \qquad h = \hat{h}(\mathbf{X}) > 0,\ T_\infty = \hat{T}_\infty(\mathbf{X}) \geqslant 0. \tag{44}$$

Another particular case arises when the body is surrounded by a vacuum. In such cases we have $h \equiv 0$ and

$$f = \varepsilon\sigma\, |T|^3\, T, \qquad \varepsilon = \hat{\varepsilon}(\mathbf{X}) > 0. \tag{45}$$

In addition to the above-mentioned situations, there exist infinitely many others in which h and/or ε are temperature dependent. For instance, when a body exchanges energy with the atmosphere by free convection, the coefficient h is given by [7, 8]

$$h = C\, |T - T_\infty|^m, \qquad C = \text{const} > 0, \tag{46}$$

in which C depends on body shape and m is such that

$$\begin{aligned} m &= \tfrac{1}{3} \qquad \text{for turbulent flow} \\ m &= \tfrac{1}{4} \qquad \text{for laminar flow.} \end{aligned} \tag{47}$$

In such cases, neglecting the radiative loss, f is given by

$$f = C\,|T - T_\infty|^m\,(T - T_\infty) \tag{48}$$

and, obviously, satisfies (11), (12), (13), and (14).

The cases in which the emittance ε depends on the temperature are also frequently found. For instance, when the considered body is metallic, the emittance is an increasing function of the temperature and, hence,

$$f = \varepsilon\sigma\,|T|^3\,T, \qquad \varepsilon = \hat{\varepsilon}(T, \mathbf{X}) \tag{49}$$

satisfies (11), (12), (13), and (14).

Situations in which the body boundary has insulated subsets are particular cases of (9), too. Such cases arise, usually, when a symmetry exists.

AN APPLICATION

We shall consider now the energy transfer phenomenon in an opaque and rigid cylindrical body, with radius R and length $2L$, as shown in Fig. 1. It will be assumed that $\mathbf{K}$, ε, h, q''', c, and T_∞ are constant, being $q''' > 0$, $T_\infty \equiv 0$, $c \equiv 0$,

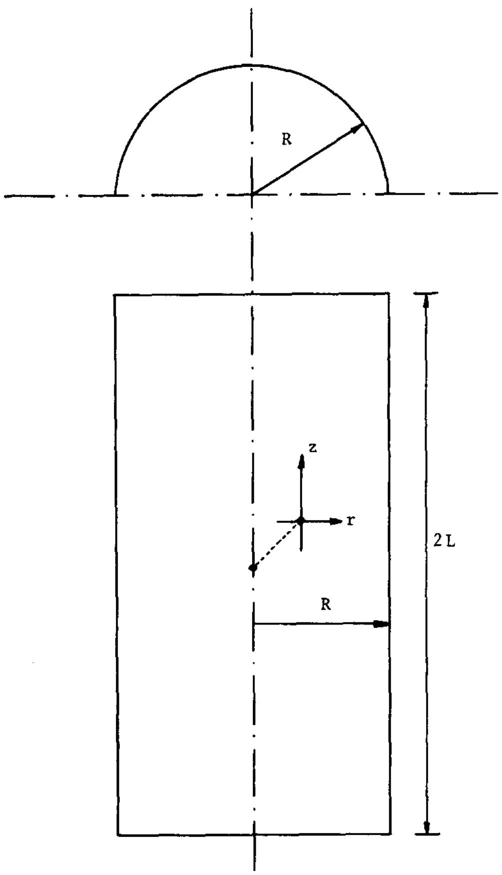

FIG. 1. The considered cylindrical body.

and $\mathbf{K} = k\mathbf{1}$. This phenomenon is mathematically described by

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{q'''}{k} = 0,\\
&\qquad 0 \leqslant r < R,\ -L < z < L,\\
&-k\frac{\partial T}{\partial r} = hT + \varepsilon\sigma\,|T|^3\,T \qquad \text{at} \quad r = R\\
&-k\frac{\partial T}{\partial z} = hT + \varepsilon\sigma\,|T|^3\,T \qquad \text{at} \quad z = L\\
&k\frac{\partial T}{\partial z} = hT + \varepsilon\sigma\,|T|^3\,T \qquad \text{at} \quad z = -L,
\end{aligned} \tag{50}$$

in which r is the radial variable and z is the axial variable.

Since a symmetry exists we shall consider, instead of (50), the description

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{q'''}{k} = 0,\\
&\qquad 0 \leqslant r < R,\ 0 < z < L,\\
&-k\frac{\partial T}{\partial r} = hT + \varepsilon\sigma\,|T|^3\,T \qquad \text{at} \quad r = R\\
&-k\frac{\partial T}{\partial z} = hT + \varepsilon\sigma\,|T|^3\,T \qquad \text{at} \quad z = L\\
&\frac{\partial T}{\partial z} = 0 \qquad\qquad\qquad \text{at} \quad z = 0.
\end{aligned} \tag{51}$$

The functional I associated to (51) is given by

$$\begin{aligned}
I[u] = {} & \frac{1}{2}\int_0^R \int_0^L k\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] r\,dz\,dr\\
& - \int_0^R \int_0^L q''' u r\,dz\,dr\\
& + \left[\frac{1}{5}\int_0^R \varepsilon\sigma\,|u|^5\,r\,dr\right]_{z=L}\\
& + \left[\frac{1}{2}\int_0^R h u^2 r\,dr\right]_{z=L}\\
& + \left[\frac{1}{5}\int_0^L \varepsilon\sigma\,|u|^5\,r\,dz\right]_{r=R}\\
& + \left[\frac{1}{2}\int_0^L h u^2 r\,dz\right]_{r=R}.
\end{aligned} \tag{52}$$

Now defining the dimensionless variables θ, ξ, and η as

$$\theta = \left[\frac{k}{q'''L^2}\right] T, \quad \xi = \frac{r}{L}, \quad \text{and} \quad \eta = \frac{z}{L} \tag{53}$$

and the dimensionless parameters α, β, and γ as

$$\alpha = \left[\frac{\varepsilon\sigma(q'''L^2/k)^4}{q'''L}\right], \beta = \frac{hL}{k}, \quad \text{and} \quad \gamma = \frac{R}{L} \tag{54}$$

we may rewrite (51) as

$$\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\theta}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{\partial\theta}{\partial\eta}\right) + 1 = 0,$$
$$0 \leqslant \xi < \gamma, 0 < \eta < 1,$$

$$\begin{aligned} -\frac{\partial\theta}{\partial\xi} &= \beta\theta + \alpha\,|\theta|^3\,\theta \quad &\text{at} \quad \xi = \gamma \\ -\frac{\partial\theta}{\partial\eta} &= \beta\theta + \alpha\,|\theta|^3\,\theta \quad &\text{at} \quad \eta = 1 \\ \frac{\partial\theta}{\partial\eta} &= 0 \quad &\text{at} \quad \eta = 0. \end{aligned} \tag{55}$$

The functional I, with the new definitions, becomes

$$\begin{aligned} I[u] = {} & \frac{1}{2}\int_0^\gamma\int_0^1\left[\left(\frac{\partial u}{\partial\xi}\right)^2 + \left(\frac{\partial u}{\partial\eta}\right)^2\right]\xi\,d\eta\,d\xi \\ & - \int_0^\gamma\int_0^1 u\xi\,d\eta\,d\xi + \left[\frac{\alpha}{5}\int_0^\gamma |u|^5\,\xi\,d\xi\right]_{\eta=1} \\ & + \left[\frac{\beta}{2}\int_0^\gamma u^2\xi\,d\xi\right]_{\eta=1} + \left[\frac{\alpha}{5}\int_0^1 |u|^5\,\xi\,d\eta\right]_{\xi=\gamma} \\ & + \left[\frac{\beta}{2}\int_0^1 u^2\xi\,d\eta\right]_{\xi=\gamma}. \end{aligned} \tag{56}$$

In order to present some results, we shall consider the following finite element approximation (see Fig. 2)

$$\begin{aligned} u = {} & (\theta_{i+1} - \theta_i)\left(\frac{\xi - \xi_j}{\Delta\xi}\right) \\ & + (\theta_i - \theta_{i+1+M})\left(\frac{\eta - \eta_j}{\Delta\eta}\right) + \theta_{i+1+M}, \\ & 1 \geqslant \frac{\eta - \eta_j}{\Delta\eta} \geqslant \frac{\xi - \xi_j}{\Delta\xi} \\ u = {} & (\theta_{i+2+M} - \theta_{i+1+M})\left(\frac{\xi - \xi_j}{\Delta\xi}\right) \\ & + (\theta_{i+1} - \theta_{i+2+M})\left(\frac{\eta - \eta_j}{\Delta\eta}\right) + \theta_{i+1+M}, \\ & \frac{\xi - \xi_j}{\Delta\xi} \geqslant \frac{\eta - \eta_j}{\Delta\eta} \geqslant 0, \qquad j = 1, 2, 3, 4, 5, ..., NM, \end{aligned} \tag{57}$$

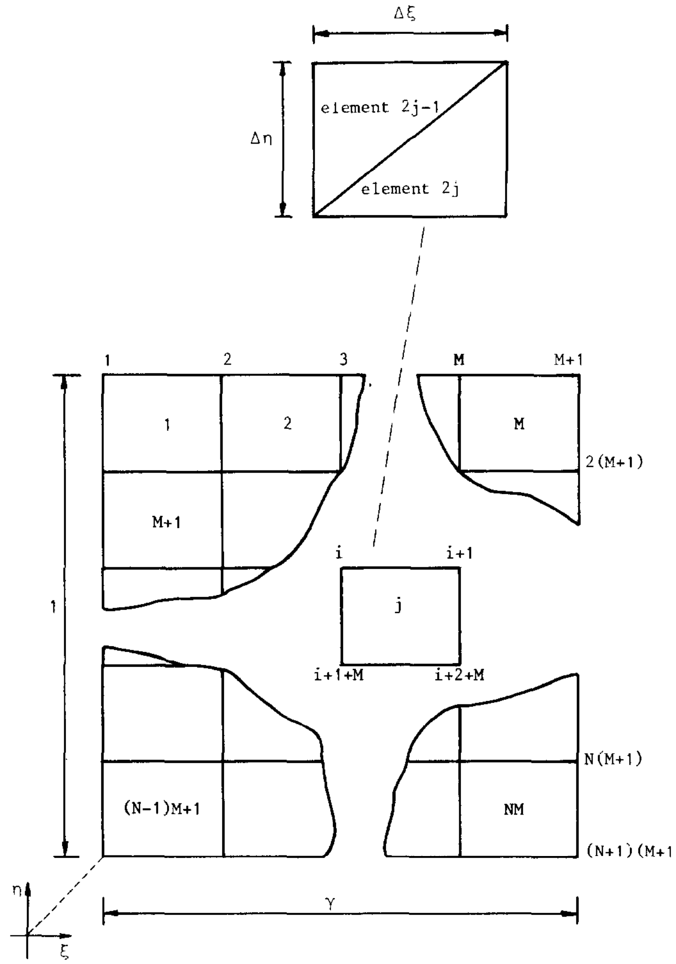

FIG. 2. The finite element approximation.

in which

$$\xi_j = \left\{j - 1 - M\left[\operatorname{int}\left(\frac{j-1}{M}\right)\right]\right\}\Delta\xi \tag{58}$$

$$\eta_j = 1 - \left\{1 + \operatorname{int}\left(\frac{j-1}{M}\right)\right\}\Delta\eta \tag{59}$$

$$i = j + \operatorname{int}\left(\frac{j-1}{M}\right). \tag{60}$$

In Eq. (58), (59), and (60), "int( )" denotes the "integer part of."

The integers M and N are such that

$$\begin{aligned} M + 1 &\equiv \text{number of nodes along the } \xi\text{-direction}, \\ N + 1 &\equiv \text{number of nodes along the } \eta\text{-direction}, \end{aligned} \tag{61}$$

and the increments $\Delta\xi$ and $\Delta\eta$ are given by

$$\Delta\xi = \gamma/M \quad \text{and} \quad \Delta\eta = 1/N. \tag{62}$$

Substituting (57) into (56) and carrying out the integrations, the functional I becomes the function

$$\phi=\hat{\phi}(\theta_1,\theta_2,\ldots,\theta_{(N+1)(M+1)})=\sum_{j=1}^{NM}(A_j+B_j)+\sum_{j=1}^{M}(C_j+D_j)+\sum_{k=1}^{N}(E_k+F_k) \tag{63}$$

in which A_j, B_j, C_j, D_j, E_k, and F_k are given as

$$A_j=\frac{1}{2}\Delta\xi\,\Delta\eta\left\{\left[\left(\frac{\theta_{i+1}-\theta_i}{\Delta\xi}\right)^2+\left(\frac{\theta_i-\theta_{i+1+M}}{\Delta\eta}\right)^2\right]\left(\frac{1}{2}\xi_j+\frac{1}{6}\Delta\xi\right)+\left[\left(\frac{\theta_{i+2+M}-\theta_{i+1+M}}{\Delta\xi}\right)^2+\left(\frac{\theta_{i+1}-\theta_{i+2+M}}{\Delta\eta}\right)^2\right]\times\left(\frac{1}{2}\xi_j+\frac{1}{3}\Delta\xi\right)\right\},\quad j=1,2,3,4,5,\ldots,NM \tag{64}$$

$$B_j=-\Delta\xi\,\Delta\eta\left\{(\theta_{i+1}-\theta_i)\left(\frac{1}{6}\xi_j+\frac{1}{12}\Delta\xi\right)+(\theta_i-\theta_{i+1+M})\left(\frac{1}{3}\xi_j+\frac{1}{8}\Delta\xi\right)+\theta_{i+1+M}\left(\xi_j+\frac{1}{2}\Delta\xi\right)+(\theta_{i+1}-\theta_{i+2+M})\left(\frac{1}{6}\xi_j+\frac{1}{8}\Delta\xi\right)+(\theta_{i+2+M}-\theta_{i+1+M})\left(\frac{1}{3}\xi_j+\frac{1}{4}\Delta\xi\right)\right\},$$
$$j=1,2,3,4,5,\ldots,NM \tag{65}$$

$$C_j=\frac{\alpha}{5}\Delta\xi\left\{\xi_j\left(\frac{|\theta_{j+1}|^5\theta_{j+1}-|\theta_j|^5\theta_j}{6(\theta_{j+1}-\theta_j)}\right)+\Delta\xi\left(\frac{|\theta_{j+1}|^7-|\theta_j|^7}{7(\theta_{j+1}-\theta_j)^2}-\frac{|\theta_{j+1}|^5\theta_{j+1}-|\theta_j|^5\theta_j}{6(\theta_{j+1}-\theta_j)^2}\theta_j\right)\right\},\quad \text{if}\quad \theta_j\neq\theta_{j+1}$$

$$C_j=\frac{\alpha}{5}(2\xi_j+\Delta\xi)\frac{\Delta\xi}{2}|\theta_j|^5,$$
$$\text{if}\quad \theta_j=\theta_{j+1},\, j=1,2,3,4,5,\ldots,M \tag{66}$$

$$D_j=\frac{\beta}{2}\Delta\xi\left\{\xi_j\left(\frac{(\theta_{j+1})^3-(\theta_j)^3}{3(\theta_{j+1}-\theta_j)}\right)+\Delta\xi\left(\frac{(\theta_{j+1})^4-(\theta_j)^4}{4(\theta_{j+1}-\theta_j)^2}-\frac{(\theta_{j+1})^3-(\theta_j)^3}{3(\theta_{j+1}-\theta_j)^2}\theta_j\right)\right\},\quad \text{if}\quad \theta_j\neq\theta_{j+1}$$

$$D_j=\frac{\beta}{2}(2\xi_j+\Delta\xi)\frac{\Delta\xi}{2}\theta_j^2,\qquad \text{if}\quad \theta_j=\theta_{j+1},$$
$$j=1,2,3,4,5,\ldots,M \tag{67}$$

$$E_k=\frac{\alpha}{5}\gamma\,\Delta\eta\left\{\frac{\left(|\theta_{k(M+1)}|^5\theta_{k(M+1)}-|\theta_{(k+1)(M+1)}|^5\theta_{(k+1)(M+1)}\right)}{6(\theta_{k(M+1)}-\theta_{(k+1)(M+1)})}\right\},$$
$$\text{if}\quad \theta_{k(M+1)}\neq\theta_{(k+1)(M+1)}$$

$$E_k=\frac{\alpha}{5}\gamma\,\Delta\eta\,|\theta_{k(M+1)}|^5,$$
$$\text{if}\quad \theta_{k(M+1)}=\theta_{(k+1)(M+1)},$$
$$k=1,2,3,4,5,\ldots,N \tag{68}$$

$$F_k=\frac{\beta}{2}\gamma\,\Delta\eta\left\{\frac{(\theta_{k(M+1)})^3-(\theta_{(k+1)(M+1)})^3}{3(\theta_{k(M+1)}-\theta_{(k+1)(M+1)})}\right\},$$
$$\text{if}\quad \theta_{k(M+1)}\neq\theta_{(k+1)(M+1)}$$

$$F_k=\frac{\beta}{2}\gamma\,\Delta\eta\,\theta^2_{k(M+1)},\qquad \text{if}\quad \theta_{k(M+1)}=\theta_{(k+1)(M+1)},$$
$$k=1,2,3,4,5,\ldots,N. \tag{69}$$

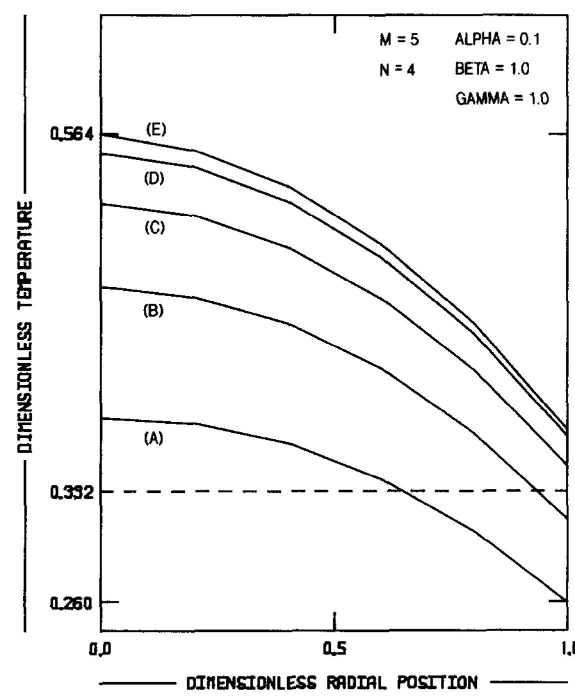

FIG. 3. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta=1.0$ (A), $\eta=0.75$ (B), $\eta=0.5$ (C), $\eta=0.25$ (D), and $\eta=0.0$ (E), obtained with 40 finite elements ($M=5$, $N=4$), assuming that $\alpha=0.1$, $\beta=1.0$, and $\gamma=1.0$. The dashed line represents the uniform temperature approximation.

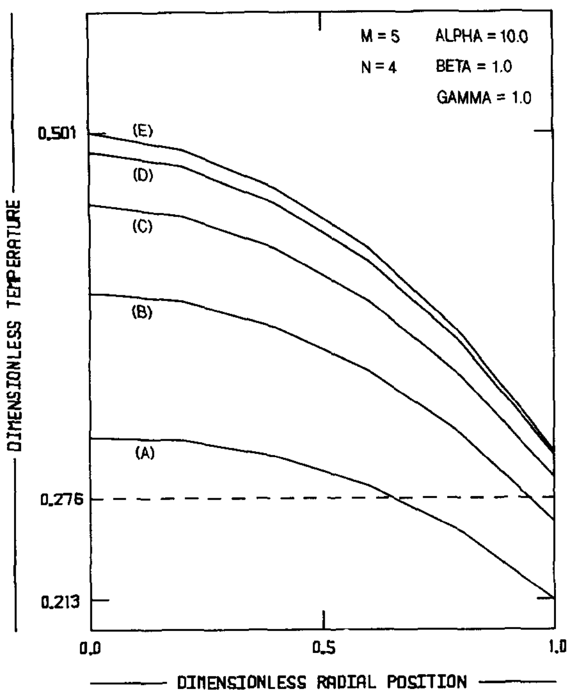

FIG. 4. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.75$ (B), $\eta = 0.5$ (C), $\eta = 0.25$ (D), and $\eta = 0.0$ (E) obtained with 40 finite elements ($M = 5$, $N = 4$), assuming that $\alpha = 10.0$, $\beta = 1.0$, and $\gamma = 1.0$. The dashed line represents the uniform temperature approximation.

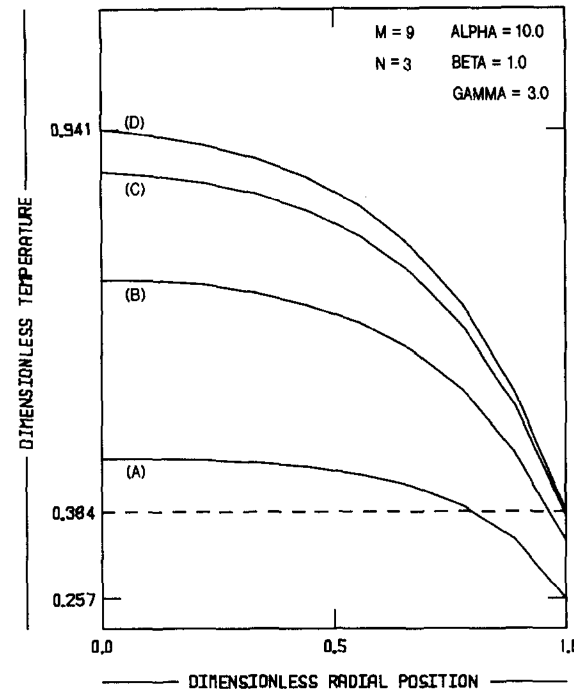

FIG. 6. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.67$ (B), $\eta = 0.33$ (C), and $\eta = 0.0$ (D) obtained with 54 finite elements ($M = 9$, $N = 3$), assuming that $\alpha = 10.0$, $\beta = 1.0$, and $\gamma = 3.0$. The dashed line represents the uniform temperature approximation.

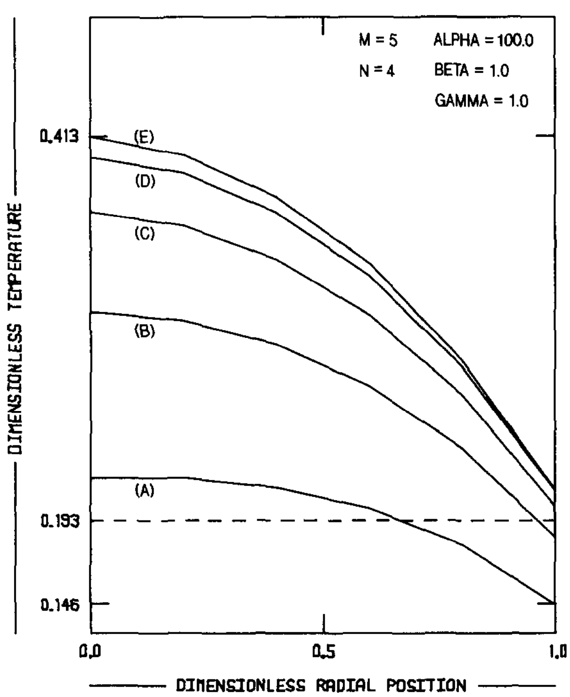

FIG. 5. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.75$ (B), $\eta = 0.5$ (C), $\eta = 0.25$ (D), and $\eta = 0.0$ (E) obtained with 40 finite elements ($M = 5$, $N = 4$), assuming that $\alpha = 100.0$, $\beta = 1.0$, and $\gamma = 1.0$. The dashed line represents the uniform temperature approximation.

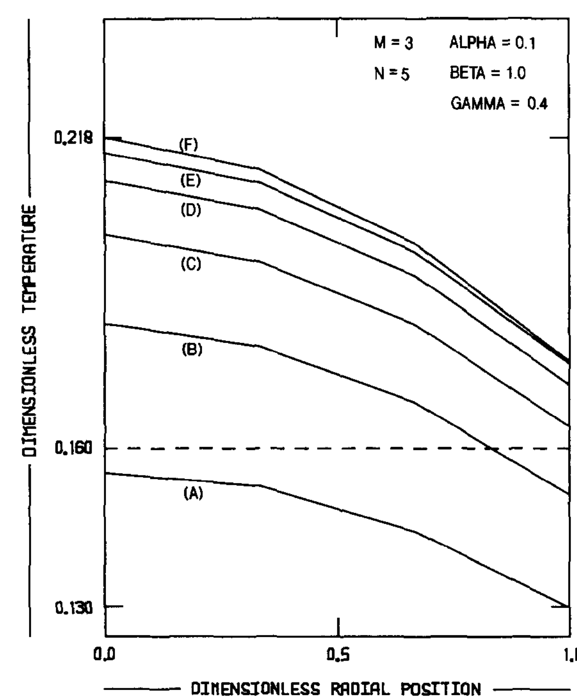

FIG. 7. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.8$ (B), $\eta = 0.6$ (C), $\eta = 0.4$ (D), $\eta = 0.2$ (E), and $\eta = 0.0$ (F) obtained with 30 finite elements ($M = 3$, $N = 5$), assuming that $\alpha = 10.0$, $\beta = 1.0$, and $\gamma = 0.4$. The dashed line represents the uniform temperature approximation.

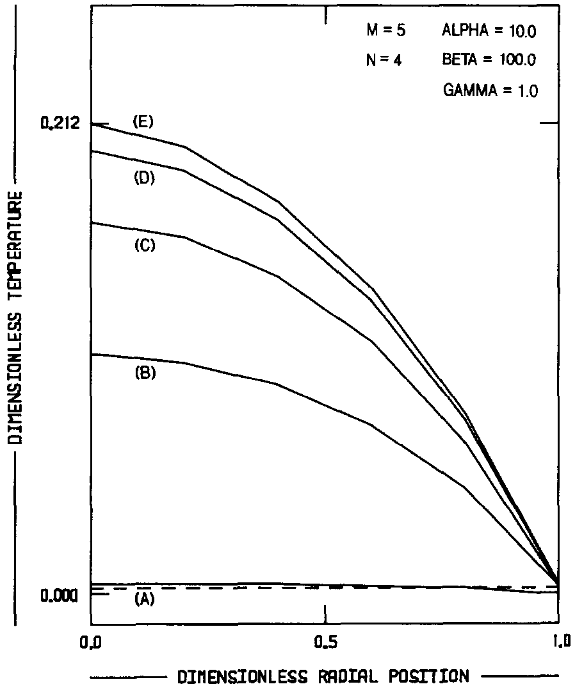

FIG. 8. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.75$ (B), $\eta = 0.5$ (C), $\eta = 0.25$ (D), and $\eta = 0.0$ (E) obtained with 40 finite elements ($M = 5$, $N = 4$), assuming that $\alpha = 10.0$, $\beta = 100.0$, and $\gamma = 1.0$. The dashed line represents the uniform temperature approximation.

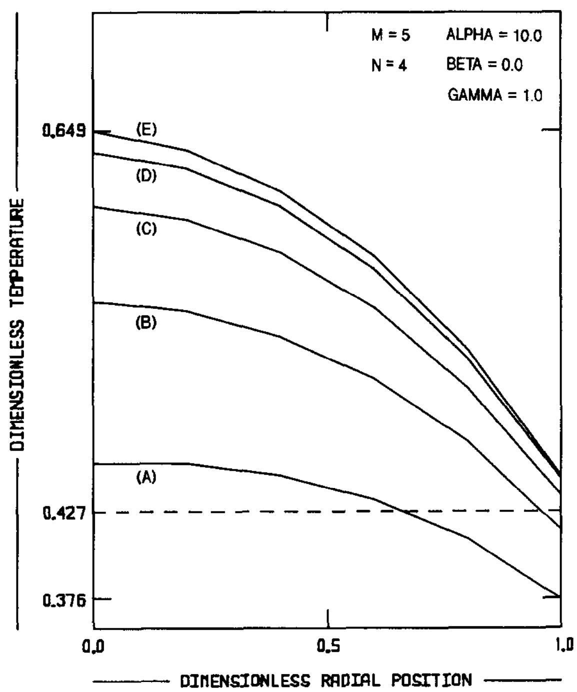

FIG. 9. The dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $\eta = 1.0$ (A), $\eta = 0.75$ (B), $\eta = 0.5$ (C), $\eta = 0.25$ (D), and $\eta = 0.0$ (E) obtained with 40 finite elements ($M = 5$, $N = 4$) assuming that $\alpha = 10.0$, $\beta = 0.0$ (without convective losses), and $\gamma = 1.0$. The dashed line represents the uniform temperature approximation.

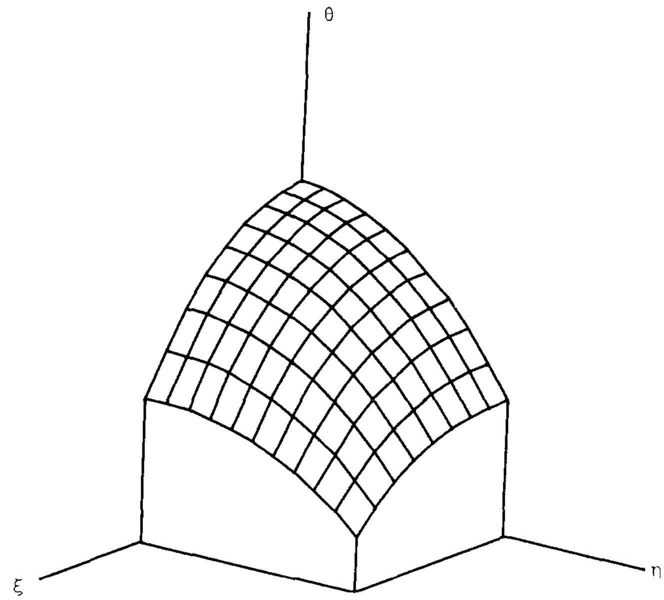

FIG. 10. The dimensionless temperature θ as a function of η and ξ obtained with 160 finite elements ($M = 10$, $N = 8$) assuming that $\alpha = 10.0$, $\beta = 1.0$, and $\gamma = 1.0$.

The relation between i and j in Eq. (64) and (65) is given by (60).

Figures 3–9 present some results obtained from the minimization of the function ϕ. The minimum was reached with the aid of the procedure presented in Appendix I. Each figure presents the dimensionless temperature θ versus the dimensionless radial position ξ/γ (or r/R) for $(N+1)$ values of the variable η.

The dashed line represents the result obtained when the temperature is assumed to be a constant. In this case we have

$$\theta = A = \text{const}, \qquad 0 \leqslant \xi \leqslant \gamma, 0 \leqslant \eta \leqslant 1 \tag{70}$$

in which A is the unique root of

$$\gamma = (\beta A + \alpha\, |A|^3\, A)(2 + \gamma). \tag{71}$$

Figure 10 presents a tridimensional representation of the solution for the particular situation considered in Fig. 4.

FINAL REMARKS

The main contribution of this work was the construction of a minimum principle suitable for nonlinear (conduction/convection/radiation) heat transfer problems and its employment in the simulation of a typical problem using finite elements.

It should be noted that the employment of (7) instead of (5) was fundamental to constructing the minimum principle and for ensuring existence and uniqueness of the solution.

When, instead of (8), we employ the description (6), the existence and the uniqueness are not ensured. This fact may be illustrated from the particular case in which the body is a sphere with unitary radius and (in a given system of units) we have $\varepsilon = 1$, $h = \sigma$, $T_\infty = 0$, $c = 0$, $q''' = \sigma$, and $\mathbf{K} = \sigma \mathbf{1}$. In such a situation, (6) reduces to

$$\begin{aligned} &\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + 1 = 0, \qquad 0 \leqslant r < 1 \\ &-\frac{dT}{dr} = T^4 + T, \qquad r = 1 \end{aligned} \tag{72}$$

in which r is the radial variable ($r = (x^2 + y^2 + z^2)^{1/2}$).

The solution (solutions) of problem (72) is given by

$$T = A - \frac{r^2}{6}, \qquad 0 \leqslant r \leqslant 1, \tag{73}$$

in which A is a (real) root of

$$\tfrac{1}{3} = (A - \tfrac{1}{6})^4 + (A - \tfrac{1}{6}). \tag{74}$$

Equation (74) admits the real roots $A = 0.489181$ and $A = -0.926131$. Thus the solution of (72) is not unique.

Now, employing (8), we have, instead of (72), the problem

$$\begin{aligned} &\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) + 1 = 0, \qquad 0 \leqslant r < 1, \\ &-\frac{dT}{dr} = |T|^3\, T + T, \qquad r = 1. \end{aligned} \tag{75}$$

The solution to (75) is also given by (73) but, now, the constant A is a root of

$$\tfrac{1}{3} = |(A - \tfrac{1}{6})|^3\,(A - \tfrac{1}{6}) + (A - \tfrac{1}{6}). \tag{76}$$

Equation (76) admits only the root $A = 0.489181$ and, hence, the solution is unique and coincides with the one (of (72)) with physical sense.

In order to present a situation in which there is no solution it is sufficient to consider the same set Ω (defined in (72)) with $\varepsilon = 1$, $h = 0$, $c = 0$, $q''' = -\sigma$, and $k = \sigma$. Under these assumptions problem (6) is given as

$$\begin{aligned} &\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) - 1 = 0, \qquad 0 \leqslant r < 1, \\ &-\frac{dT}{dr} = T^4, \qquad r = 1. \end{aligned} \tag{77}$$

The solution of problem (77) is given by

$$T = A + \frac{r^2}{6}, \qquad 0 \leqslant r < 1 \tag{78}$$

in which A is a (real) root of

$$-\tfrac{1}{3} = (A + \tfrac{1}{6})^4. \tag{79}$$

It is obvious that (79) does not admit real roots. Thus, there exist no real solution in this case.

APPENDIX I: THE MINIMIZATION OF ϕ

The minimum of the function ϕ, defined by (63), is reached through an iterative process in which each step consists of minimizing a function of one real variable. This process may be summarized as

(1) The unknown θ_i ($i = 1, 2, \ldots, (N+1)(M+1)$) are initialized as

$$\theta_i^0 = A; \qquad i = 1, 2, \ldots, (N+1)(M+1), \tag{AI.1}$$

in which A is the root of (71).

(2) Minimizing each function ϕ_i ($\phi_i = \hat{\phi}_i(\theta_i)$) given by

$$\begin{aligned} \phi_1 &= \hat{\phi}_1(\theta_1) = \hat{\phi}(\theta_1, \theta_2^0, \theta_3^0, \ldots, \theta^0_{(N+1)(M+1)}) \\ \phi_2 &= \hat{\phi}_2(\theta_2) = \hat{\phi}(\theta_1^1, \theta_2, \theta_3^0, \ldots, \theta^0_{(N+1)(M+1)}) \\ \phi_3 &= \hat{\phi}_3(\theta_3) = \hat{\phi}(\theta_1^1, \theta_2^1, \theta_3, \theta_4^0, \ldots, \theta^0_{(N+1)(M+1)}) \\ &\;\vdots \\ \phi_{(N+1)(M+1)} &= \hat{\phi}_{(N+1)(M+1)}(\theta_{(N+1)(M+1)}) \\ &= \hat{\phi}(\theta_1^1, \theta_2^1, \theta_3^1, \ldots, \theta^1_{(N+1)(M+1)-1}, \\ &\qquad \theta_{(N+1)(M+1)}), \end{aligned} \tag{AI.2}$$

in which θ_i^1 is obtained from the minimization of $\hat{\phi}_i(\theta_i)$.

(3) Having obtained the θ_i^1's we may repeat the second step for obtaining the θ_i^2's. This procedure is repeated until the convergence is reached.

The minimum of each function ϕ_i corresponds to the root of its first derivative ϕ_i'. This root is, at the iteration $n+1$, reached with the following (Newton–Raphson) scheme

$$x^{m+1} = x^m - \hat{\phi}_i'(x^m)/\hat{\phi}_i''(x^m) \tag{AI.3}$$

in which

$$x^0 = \theta_i^n (\text{root of } \hat{\phi}_i'(\theta_i) = 0 \text{ at the } n\text{th iteration}) \tag{AI.4}$$

and

$$\theta_i^{n+1} = x^{m'}, \tag{AI.5}$$

m' being chosen in such a way that, for a given small δ,

$$\left| \frac{x^{m'+1} - x^{m'}}{x^{m'}} \right| < \delta. \tag{AI.6}$$

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